The Necessary and Sufficient Conditions of Separability for Bipartite Pure States in Infinite Dimensional Hilbert Spaces

Su Hu* and Zongwen Yu†
Department of Mathematical Sciences, Tsinghua University, Beijing, 100084 China

In this letter, we present the necessary and sufficient conditions of separability for bipartite pure states in infinite dimensional Hilbert spaces. Let M be the matrix of the amplitudes of $|\psi\rangle$, we prove M is a compact operator. We also prove $|\psi\rangle$ is separable if and only if M is a bounded linear operator with rank 1, that is the image of M is a one dimensional Hilbert space. So we have related the separability for bipartite pure states in infinite dimensional Hilbert spaces to an important class of bounded linear operators in Functional analysis which has many interesting properties.

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A pure state is separable if and only if it can be written as a tensor product of states of different subsystems. It is also known that a state $|\psi\rangle$ of a bipartite system is separable if and only if it has Schmidt number 1 [1]. Peres presented a necessary and sufficient condition for the occurrence of Schmidt decomposition for a tripartite pure state [2] and showed that the positivity of the partial transpose of a density matrix is a necessary condition for separability [3]. Thapliyal showed that a multipartite pure state is Schmidt decomposable if and only if the density matrices obtained by tracing out any party are separable [4]. In [5], Dafa Li obtained a necessary and sufficient conditions of separability for multipartite pure state in finite dimensional Hilbert spaces. In [6], Yu and Hu gave other necessary and sufficient conditions, and simplified the proof of the main result in [5]. They also obtained an algorithm to determine the separability of any multipartite pure state efficiently and quickly. But all of the previous works only aimed to determine the multipartite pure state in finite dimensional Hilbert spaces. In this letter, we gave the necessary and sufficient conditions of separability for bipartite pure state in infinite dimensional Hilbert spaces and relate this problem to an important class of bounded linear operators in Functional analysis.

Let H_A, H_B be two infinite dimensional Hilbert spaces. We define their tensor product as in [7]. Let $H_A \overset{\otimes}{a} H_B$ denote the algebraic tensor product of H_A and H_B consider as a linear space over $\mathbb{C}$. It is easy to see $(\sum_{i=1}^{m} h_i \otimes k_i, \sum_{j=1}^{n} h_j^{'} \otimes k_j^{'}) = \sum_{i=1}^{m}\sum_{j=1}^{n} (h_i, h_j^{'})(k_i, k_j^{'})$ defines an inner product of $H_A \overset{\otimes}{a} H_B$. We should note that this space is not complete with this inner product. Pass to completion, we get a Hilbert space. As in Functional analysis, we call it the tensor product of H_A and H_B and denote it by $H_A \otimes H_B$. We can prove [7], if $\{e_\alpha\}_{\alpha\in A}$ and $\{f_\beta\}_{\beta\in B}$ are orthonormal basis for H_A and H_B respectively, then $\{e_\alpha \otimes f_\beta\}_{(\alpha,\beta)} \in H_A \otimes H_B$ is an orthonormal basis for $H_A \otimes H_B$.

Let us now consider two physical systems A and B represented by the Hilbert spaces H_A and H_B respectively. The joint system is represented by the Hilbert space $H_A \otimes H_B$. Let $|\psi\rangle \in H_A \otimes H_B$ be a pure state of a composite system AB. We also say $|\psi\rangle$ is separable if and only if it can be written as a tensor product of states of different subsystems. In this letter, we suppose H_A and H_B have countable number of dimensions, this is equivalent to say that H_A and H_B are separable topological spaces [8]. Let $|i\rangle(|j\rangle)$ be the orthonormal basis for Hilbert space $H_A(H_B)$. From above, we know $|i\rangle|j\rangle$ is an orthonormal basis for Hilbert space $H_A \otimes H_B$. Since $|\psi\rangle \in H_A \otimes H_B$, we can give the Fourier expansion of $|\psi\rangle$ under this basis. Then we can write $|\psi\rangle = \sum_{i,j} a_{ij}|i\rangle|j\rangle$, where $a_{ij} \in \mathbb{C}$ and $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} |a_{ij}|^2 = 1$. Let $M = (a_{ij})$ be the infinite (but countable) dimensional matrix of the amplitudes of $|\psi\rangle$. We will prove M is a compact linear operator in Functional analysis and give the criterion for the separability.

Definition 1. [8] *l^2 denote the set of all infinite sequences $\{x_n\}_{n=1}^{\infty}$ such that $\sum_{i=1}^{\infty} |x_i|^2 < \infty$. For x, y in l^2 define the inner product by $(x, y) = \sum_{i=1}^{\infty} x_i \bar{y_j}$. It is easy to show l^2 is a Hilbert space with this inner product.*

For $x = \{x_n\}_{n=1}^{\infty} \in l^2$, denote $x = (x_1, x_2, \cdots, x_n, \cdots)^T, \bar{x} = (\bar{x_1}, \bar{x_2}, \cdots, \bar{x_n}, \cdots)^T$.

Theorem 1. *$|\psi\rangle$ is separable if and only if there exist two unit vectors $x, y \in l^2$ such that $M = xy^\dagger$.*

Proof. $\Rightarrow$ By definition, $|\psi\rangle$ is separable if and only if we can write $|\psi\rangle = (\sum_{i=1}^{\infty} x_i|i\rangle) \otimes \left(\sum_{j=1}^{\infty} y_j|j\rangle\right)$ where $\sum_{i=1}^{\infty} |x_i|^2 = 1$ and $\sum_{j=1}^{\infty} |y_j|^2 = 1$. As above $|\psi\rangle = \sum_{i,j} a_{ij}|i\rangle|j\rangle$ is the Fourier expansion of $|\psi\rangle$ under the orthonormal basis $|i\rangle|j\rangle$ in $H_A \otimes H_B$. From the definition of tensor product for infinite dimensional Hilbert spaces.

We have

$$\begin{aligned} a_{ij} &= (|\psi\rangle, |i\rangle|j\rangle) \\ &= \left(\left(\sum_{i'=1}^{\infty} x_{i'}|i^{'}\rangle\right) \otimes \left(\sum_{j'=1}^{\infty} y_{j'}|j^{'}\rangle\right), |i\rangle|j\rangle\right) \\ &= \left(\sum_{i'=1}^{\infty} x_{i'}|i^{'}\rangle, |i\rangle\right)\left(\sum_{j'=1}^{\infty} y_{j'}|j^{'}\rangle, |j\rangle\right) \\ &= x_i y_j \end{aligned} \tag{1}$$

We set $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ \vdots \end{pmatrix}, y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \\ \vdots \end{pmatrix}$. From $\sum_{i=1}^{\infty} |x_i|^2 = 1$ and $\sum_{j=1}^{\infty} |y_j|^2 = 1$, we see $x, y \in l^2$ and as above $M = (a_{ij}) = \begin{pmatrix} a_{11} & \cdots & a_{1n} & \cdots \\ \vdots & \ddots & \vdots & \ddots \\ a_{n1} & \cdots & a_{nn} & \cdots \\ \vdots & \ddots & \vdots & \ddots \end{pmatrix}$. From (1) and the multiplication law for infinite (countable) dimensional matrices, we see $M = xy^{\dagger}$.

$\Leftarrow$ Suppose that $M = xy^{\dagger}, x, y \in l^2$. Because $M = (a_{ij})$ be the matrix of the amplitudes of $|\psi\rangle$, we know $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} |a_{ij}|^2 = 1$. From $M = xy^{\dagger}$ and the multiplication for infinite (countable) dimensional matrices, we see $a_{ij} = x_i \bar{y_j}$. From $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} |a_{ij}|^2 = 1$, we have $\sum_{j=1}^{\infty} |y_j|^2 \sum_{i=1}^{\infty} |x_i|^2 = 1$. We denote $\parallel x \parallel = \sum_{i=1}^{\infty} |x_i|^2 < +\infty, \parallel y \parallel = \sum_{j=1}^{\infty} |y_j|^2 < +\infty$ (because $x, y \in l^2$). Under the suppose $\widetilde{x} = \frac{x}{\|x\|}, \widetilde{y} = \frac{y}{\|y\|}$, we also have $M = \widetilde{x}\widetilde{y}$. So we can suppose $\parallel x \parallel = \parallel y \parallel = 1$ and construct two states $|v\rangle = \sum_{i=1}^{\infty} x_i|i\rangle, |w\rangle = \sum_{j=1}^{\infty} \bar{y_j}|j\rangle$. We have

$$\begin{aligned} |v\rangle \otimes |w\rangle &= \left(\sum_{i=1}^{\infty} x_i|i\rangle\right) \otimes \left(\sum_{j=1}^{\infty} y_j|j\rangle\right) \\ &= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} x_i\bar{y_j}|i\rangle|j\rangle = \sum_{i,j} a_{ij}|i\rangle|j\rangle = |\psi\rangle \end{aligned}$$

We see $|\psi\rangle$ is a separable pure state. □

Lemma 1. *If M is the matrix of the amplitudes of a pure state $|\psi\rangle$, then $M = xy^{\dagger}$ if and only if the determinants of all the 2×2 submatrices of M are zero.*

Proof. $\Rightarrow M = xy^{\dagger}$, where $M = (a_{ij}), x = (x_i), y = (y_j)$. As above we see $a_{ij} = x_i\widetilde{y_j}$. $m = \begin{pmatrix} a_{il} & a_{ik} \\ a_{jl} & a_{jk} \end{pmatrix}$ is any 2×2 submatrix of M. It is easy to check $det(m) = a_{il}a_{jk} - a_{ik}a_{jl} = x_i\bar{y_l}x_j\bar{y_k} - x_i\bar{y_k}x_j\bar{y_l} = 0$. Therefore if $|\psi\rangle$ is separable, the determinates of all the 2×2 submatrices are zero.

$\Leftarrow$ Suppose $M = (M_1, M_2, \cdots, M_j, \cdots)$ (we can suppose $M_1 \neq 0$). If M_1, M_j are linearly independent for some $j > 1$, then the 2×2 submatrix is reversible, so $det(m) \neq 0$. This is a contradiction. So for each j we have a constant $\lambda_j \in \mathbb{C}$, such that $M_j = \lambda_j M_1$. Then

$$\begin{aligned} M &= (M_1, M_2, \cdots, M_j, \cdots) \\ &= (M_1, \lambda_2 M_1, \cdots, \lambda_j M_1, \cdots) \\ &= M_1(1, \lambda_2, \cdots, \lambda_j, \cdots) \end{aligned} \tag{2}$$

Set $x = M_1, y = (1, \bar{\lambda_2}, \cdots, \bar{\lambda_j}, \cdots)^T$. Since M is the matrix of the amplitudes of $|\psi\rangle$, it is obviously $x, y \in l^2$. From (2) we have $M = xy^{\dagger}$, as desired. □

Theorem 2. *$|\psi\rangle$ is a separable pure state if and only if the determinate of all the 2×2 submatrices of M are zero.*

Proof. This is immediately from theorem 1 and lemma 1. □

Remark *Theorem 2 generalize the corresponding result in [5] to infinite dimensional Hilbert spaces.*

Definition 2. [8] *An operator T on Hilbert space H has finite rank if $Im(T)$(the image of T) is finite dimensional.*

Theorem 3. *$|\psi\rangle$ is a pure state in $H_A \otimes H_B$. M is the matrix of the amplitudes of $|\psi\rangle$, then M is a compact linear operator on the Hilbert space l^2.*

Proof. 1) Denote $M = \begin{pmatrix} M_1^T \\ M_2^T \\ \vdots \\ M_j^T \\ \vdots \end{pmatrix}$. $\forall x \in l^2$ we have

$$Mx = \begin{pmatrix} M_1^T \\ M_2^T \\ \vdots \\ M_j^T \\ \vdots \end{pmatrix} x = \begin{pmatrix} M_1^T \cdot x \\ M_2^T \cdot x \\ \vdots \\ M_j^T \cdot x \\ \vdots \end{pmatrix} = \begin{pmatrix} (M_1, \bar{x}) \\ (M_2, \bar{x}) \\ \vdots \\ (M_j, \bar{x}) \\ \vdots \end{pmatrix}$$

So $\parallel Mx \parallel^2 = \sum_{i=1}^{\infty} |(M_i, \bar{x})|^2 \leq \sum_{i=1}^{\infty} \parallel M_i \parallel^2 \parallel x \parallel^2 = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} |a_{ij}|^2 \parallel x \parallel^2 = \parallel x \parallel^2$. We get M is a well defined bounded linear operator on l^2 with the norm $\|M \| \leq 1$.

2) To prove M is a compact operator, according to [8] we should only to show there is a sequence $\{T_n\}$ of operators of finite rank such that $\parallel M - T_n \parallel \to 0 (n \to \infty)$. Because $M = (a_{ij})$ is the matrix of the amplitudes of

$|\psi\rangle$, we have $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|a_{ij}|^2 = 1$. We set

$$M_n = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & 0 & 0 & \cdots \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 & 0 & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & 0 & 0 & \cdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

obviously M_n has finite rank.

Denote $s = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|a_{ij}|^2 = 1, s_n = \sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|^2$, from the absolute convergence of s, we know

$$|s - s_n| \to 0 \quad (n \to \infty). \tag{3}$$

We have

$$M - M_n = \begin{pmatrix} 0 & 0 & \cdots & 0 & a_{1(n+1)} & a_{1(n+2)} & \cdots \\ 0 & 0 & \cdots & 0 & a_{2(n+1)} & a_{2(n+2)} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots \\ 0 & 0 & \cdots & 0 & a_{n(n+1)} & a_{n(n+2)} & \cdots \\ a_{(n+1)1} & a_{(n+1)2} & \cdots & a_{(n+1)n} & a_{(n+1)(n+1)} & a_{(n+1)(n+2)} & \cdots \\ a_{(n+2)1} & a_{(n+2)2} & \cdots & a_{(n+2)n} & a_{(n+2)(n+1)} & a_{(n+2)(n+2)} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} B_1^T \\ B_2^T \\ \vdots \\ B_n^T \\ B_{n+1}^T \\ B_{n+2}^T \\ \vdots \end{pmatrix}. \tag{4}$$

From $\sum_{i,j=1}^{\infty}|a_{ij}|^2 = 1$, we see $B_i \in l^2 (i \geq 1)$.

$(l^2)_1$ denote the closed unit ball in l^2. $\forall \phi \in (l^2)_1$, We have

$$(M - M_n)\phi = \begin{pmatrix} B_1^T \\ B_2^T \\ \vdots \\ B_j^T \\ \vdots \end{pmatrix}\phi$$

$$= \begin{pmatrix} B_1^T \cdot \phi \\ B_2^T \cdot \phi \\ \vdots \\ B_j^T \cdot \phi \\ \vdots \end{pmatrix} = \begin{pmatrix} (B_1, \bar{\phi}) \\ (B_2, \bar{\phi}) \\ \vdots \\ (B_j, \bar{\phi}) \\ \vdots \end{pmatrix}. \tag{5}$$

Then $\| (M - M_n)\phi \|^2 = \sum_{i=1}^{\infty}|(B_i, \bar{\phi})|^2 \leq \sum_{i=1}^{\infty}\| B_i \|^2 \| \phi \|^2 \leq \sum_{i=1}^{\infty}\| B_i \|^2$. From (4) we get

$$\begin{aligned} \sum_{i=1}^{\infty}\| B_i \|^2 &= \sum_{i=1}^{n}\| B_i \|^2 + \sum_{i=n+1}^{\infty}\| B_i \|^2 \\ &= \sum_{i=1}^{n}\sum_{j=n+1}^{\infty}|a_{ij}|^2 + \sum_{i=n+1}^{\infty}\sum_{j=1}^{\infty}|a_{ij}|^2 \\ &= |s - s_n| \end{aligned} \tag{6}$$

From (5) and (6), $\| (M - M_n)\phi \|^2 \leq |s - s_n|, \forall \phi \in (l^2)_1$. From the definition of the norm of the operators on l^2, we have $\| M - M_n \|^2 = \sum_{\phi \in (l^2)_1} \| (M - M_n)\phi \|^2 \leq |s - s_n|$. From (3), we get $\| M - M_n \| \to 0 (n \to \infty)$. So we see M is a compact operator on l^2. □

Remark: *We know all the compact operators form a closed two sided ideal in operator algebra and they have many interesting properties. For example [8]*

(1) "If A is a compact linear operator on $H, \lambda \in \mathbb{C}$ and $\lambda \neq 0$, then the image of A is closed and $\dim\ker(A - \lambda I) = \dim\ker(A - \lambda I)^\dagger < \infty$", this is a famous theorem named "The Fredholm Alternative" in Functional analysis. Someone call this is "the linear algebra of infinite dimensional spaces".

(2) "A is compact if and only if $A^\dagger$ is compact." This is a theorem of Schaduer.

Lemma 2. *[8] If T is a positive compact operator, then there is a unique positive compact operator A such that $A^2 = T$. A is called the positive square root of T.*

Lemma 3. *[8] (Polar decomposition of compact operators.) Let T be a compact operator on Hilbert space H and let A be the unique positive square root of $T^\dagger T$. Then (a) $\| Ah \| = \| Th \|$ for all h in H. (b) There is a unique operator U such that $\| Uh \| = \| h \|$ when $h \perp \ker T, Uh = 0$, when $h \in \ker T$ and $UA = T$.*

Theorem 4. *$|\psi\rangle$ is a pure state in $H_A \otimes H_B$. M is the matrix of the amplitudes of $|\psi\rangle$, then M has polar decomposition.*

Proof. Form theorem 3 and Lemma 3. □

We will see if $|\psi\rangle$ is separable, M is not only a compact linear operator, but also an operator with rank 1. We know from Analysis the operators which have finite rank must be a compact operator [8].

Lemma 4. *If M is the matrix of the amplitudes of a pure state $|\psi\rangle$, then $M = xy^\dagger$, where $x, y \in l^2$ if and only if M is a bounded linear operator with rank 1.*

Proof. $\Rightarrow$ $\forall z \in l^2$, denote $y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \\ \vdots \end{pmatrix}, z = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \\ \vdots \end{pmatrix}$.

We have

$$Mz = xy^\dagger z = x(\bar{y_1}, \bar{y_2}, \cdots, \bar{y_n}, \cdots) \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \\ \vdots \end{pmatrix} = x\left(\sum_{i=1}^{\infty} \bar{y_i} z_i\right),$$

so $ImM \subset \text{span}(x)$. But $M \neq 0$, we get $\dim ImM = 1$.

$\Leftarrow$ Suppose $\dim ImM = 1$. Denote $M = (M_1, M_2, \cdots, M_i, \cdots)$(we can suppose $M_1 \neq 0$). Suppose the vector $e_i \in l^2$ is the vector with all 0s except for a 1 in the ith coordinate. We have $Me_1 = M_1$ and $Me_i = M_i$. Then $M_1, M_i \in ImM$, but $\dim ImM = 1$, so there exists $\lambda_i \in \mathbb{C}$ such that $M_i = \lambda_i M_1$. We have

$$\begin{aligned} M &= (M_1, M_2, \cdots, M_i, \cdots) \\ &= (M_1, \lambda_2 M_1, \cdots, \lambda_i M_1, \cdots) \\ &= M_1(1, \lambda_2, \cdots, \lambda_i, \cdots) \end{aligned} \tag{7}$$

Denote $x = M_1, y = (1, \lambda_2, \cdots, \lambda_i, \cdots)^\dagger$. M is the matrix of the amplitudes of $|\psi\rangle$, we see that $x, y \in l^2$. Finally, from (7) we get $M = xy^\dagger$, as desired. □

Theorem 5. *$|\psi\rangle$ is a separable pure state if and only if M is a bounded linear operator with rank 1.*

Proof. From theorem 1 and lemma 4. □

Corollary 1. *$|\psi\rangle$ is a separable pure state in $H_A \otimes H_B$. M is the matrix of the amplitudes of $|\psi\rangle$. We have following results 1) $ImM^\dagger$ is also closed; 2) $ImM = \ker M^{\dagger\perp}$; 3) $ImM^\dagger = \ker M^\perp$; 4) M is an open mapping; 5) $M^\dagger$ is an open mapping.*

Proof. According to theorem 5, $\dim ImM = 1$, so ImM is a closed subspace in l^2. All the above results follow from the closed range theorem [8]. □

Denote $\mathscr{L}(l^2)$ to be the algebra of bounded linear operators on l^2 and $\mathscr{B}^0(l^2)$ to be the ideal of compact operators on l^2. We also define two subsets of $\mathscr{L}(l^2)$, $\mathscr{F} = \{M \in \mathscr{L}(l^2) : M \text{ is a bounded linear operator with rank 1}\}$. $\mathscr{T} = \{M \in \mathscr{L}(l^2) : M \text{ is the matrix of a bipatite pure state of a compositesystem AB}\}$.

From theorem 3 and 5, we get $\mathscr{T} \subset \mathscr{F} \subset \mathscr{B}^0(l^2) \subset \mathscr{L}(l^2)$, and relate the separability for bipartite pure states to an important class of bounded linear operators in Functional analysis.

So, given a pure state $|\psi\rangle$ in $H_A \otimes H_B$, M is the matrix of the amplitudes of $|\psi\rangle$, if M is not a rank 1 operator on l^2, we can conclude that $|\psi\rangle$ is not separable. But the rank 1 operators in an infinite dimensional Hilbert space are rare, so the separable pure states in $H_A \otimes H_B$ are also rare.

* Electronic address: hus04@mails.tsinghua.edu.cn
† Electronic address: yzw04@mails.tsinghua.edu.cn

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